Sequences, Series, and Power Series

Definition Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \quad \text{is called an infinite series (or just a series)}$$

and

$$s_n = \sum_{k=1}^{n} a_k \quad \text{is called the } n^{\text{th}} \text{ partial sum of } \sum_{k=1}^{\infty} a_k.$$

- The series $\sum_{k=1}^{\infty} a_k$ is called convergent if the sequence $\{s_n\}$ is convergent, or equivalently $\sum_{k=1}^{\infty} a_k$ is convergent if $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k \right) = 0$.
- A number $s \in \mathbb{R}$ is called the sum of the series $\sum_{k=1}^{\infty} a_k$ if

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k \quad \text{i.e. } \sum_{k=1}^{\infty} a_k \text{ is convergent and it conveges to } s.$$

- The series is called divergent if the sequence $\{s_n\}$ is divergent.

Theorems

1. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, and if $c \in \mathbb{R}$, then so are the series

$$\sum_{k=1}^{\infty} c\, a_k, \quad \sum_{k=1}^{\infty} (a_k + b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (a_k - b_k),$$

with respectively

$$\sum_{k=1}^{\infty} c\, a_k = c \sum_{k=1}^{\infty} a_k, \quad \sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \quad \text{and} \quad \sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k.$$

2. (A Test for Divergence) If $\lim_{k\to\infty} a_k$ does not exist or if $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent. Equivalently, if the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k = 0$.
3. (Geometric Series) If $r \neq 1$ is a real number, then the geometric series

$$\sum_{k=1}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=1}^{n} r^k = \lim_{n\to\infty} \frac{r - r^{n+1}}{1-r} \begin{cases} \text{converges to } \dfrac{r}{1-r} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| > 1. \end{cases}$$

Examples

1. Suppose that

$$s_n = \sum_{k=1}^{n} a_k = \frac{2n}{3n+5} \quad \text{for each } n = 1,\, 2,\, \ldots.$$

Then

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \frac{2}{3}.$$

2. The sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = 5 \cdot \frac{1}{1-(-\frac{2}{3})} = 3.$$

3. The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent since

$$s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2+1} + \frac{1}{2^2}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots \frac{1}{2^n}\right) \geq 1 + \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{2^{n-1}}{2^n} = 1 + \frac{n}{2}.$$

4. The sum of the series $\sum_{k=1}^{\infty} \left[\frac{3}{k(k+1)} + \frac{1}{2^k}\right]$ is

$$\sum_{k=1}^{\infty} \left[\frac{3}{k(k+1)} + \frac{1}{2^k}\right] = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{3}{k} - \sum_{k=1}^{n} \frac{3}{k+1}\right] + \frac{\frac{1}{2}}{1-\frac{1}{2}} = 3 \cdot 1 + 1 = 4.$$

The Integral Test Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_k = f(k)$. Then

- the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent.

In other words,

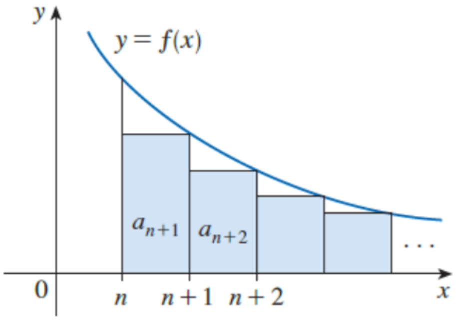
(i) if $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is convergent.

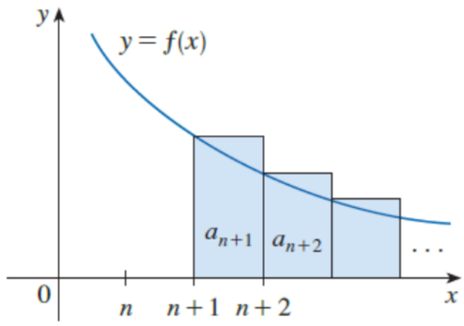
(ii) if $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{k=1}^{\infty} a_k$ is divergent.

Proof For each $k = 1,\, 2, \ldots,$ let $a_k = f(k)$ and for each $n = 1,\, 2, \ldots$ let

$$R_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = s - s_n \quad \text{be the } n^{\text{th}} \text{ remainder term.}$$

Since f is decreasing on $[1, \infty)$, we have





- $a_k = f(k) \geq f(x) \geq f(k+1) = a_{k+1}$ for each $x \in [k, k+1]$ and for each $k = 1, 2, \ldots,$
- $\displaystyle\int_n^\infty f(x)\,dx \overset{(*)}{\geq} R_n = \sum_{k=n+1}^\infty a_k = a_{n+1} + a_{n+2} + \cdots \overset{(\dagger)}{\geq} \int_{n+1}^\infty f(x)\,dx$ for each $n \geq 1$.

Thus

$$\int_1^\infty f(x)dx \text{ is convergent } \overset{\text{def}}{\iff} \lim_{n\to\infty} \int_n^\infty f(x)dx = 0,$$

$$\underset{(\dagger)}{\overset{(*)}{\iff}} \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=n+1}^\infty a_k = 0 \overset{\text{def}}{\iff} \sum_{k=1}^\infty a_k \text{ is convergent.}$$

Examples

1. The p-series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

2. Test the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ for convergence or divergence.

3. Determine whether the series $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n}$ converges or diverges.

   [Note that if $f(x) = \dfrac{\ln x}{x}$, $x \geq 1$, then $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e$, and $f(x)$ is positive, decreasing on $[e, \infty)$.]

4.
   (a) Approximate the sum of the series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
   [Solution: $s_{10} \approx 1.1975$ and since $R_{10} = s - s_{10} \overset{(*)}{\leq} \displaystyle\int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{200} = 0.005$, the size of the error is at most 0.005.]

   (b) How many terms are required to ensure that the sum is accurate to within 0.0005?
   [Solution: Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leq 0.0005$. Since $R_n \overset{(*)}{\leq} \displaystyle\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$, we want $\dfrac{1}{2n^2} < 0.0005 \implies n^2 > 1000$ or $n > \sqrt{1000} \approx 31.6$. So we need 32 terms to ensure accuracy to within 0.0005.]

5. Note that if we add s_n to each side of estimates (∗), (†) for the remainder $R_n = s - s_n$, we get a lower bound and an upper bound for s.

$$s_n + \int_n^\infty f(x)\,dx \geq s_n + R_n = s \geq s_n + \int_{n+1}^\infty f(x)\,dx \implies \int_n^\infty f(x)\,dx \geq s - s_n \geq \int_{n+1}^\infty f(x)\,dx.$$

The Comparison Tests

- (The Direct Comparison Test) Suppose that $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty b_k$ are series with positive terms.
  (a) If $\sum_{k=1}^\infty b_k = \lim_{n\to\infty} \sum_{k=1}^n b_k$ is convergent and $a_k \leq b_k$ for all k, then $\sum_{k=1}^\infty a_k$ is also convergent.
  (b) If $\sum_{k=1}^\infty b_k$ is divergent and $a_k \geq b_k$ for all k, then $\sum_{k=1}^\infty a_k = \lim_{n\to\infty} \sum_{k=1}^n a_k$ is also divergent.
- (The Limit Comparison Test) Suppose that $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty b_k$ are series with positive terms.
  (a) If $\lim_{k\to\infty} \frac{a_k}{b_k} = r \in (0, \infty)$, then either both series converge or both diverge.
  (b) If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and if $\sum_{k=1}^\infty b_k$ is convergent, then $\sum_{k=1}^\infty a_k$ is convergent.
  (c) If $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and if $\sum_{k=1}^\infty a_k$ is convergent, then $\sum_{k=1}^\infty b_k$ is convergent.

Examples

1. Determine whether the series $\sum_{k=1}^\infty \frac{5}{2k^2+4k+3}$ converges or diverges.
2. Test the series $\sum_{k=1}^\infty \frac{1}{2^k - 1}$ for convergence or divergence.
3. Determine whether the series $\sum_{k=1}^\infty \frac{2k^2+3k}{\sqrt{5+k^5}}$ converges or diverges.
4. Use the sum of the first 100 terms to approximate the sum of the series $\sum_{k=1}^\infty \frac{1}{k^3+1}$. Estimate the error involved in this approximation.
   [Solution: Let

$$R_n = \sum_{k=n+1}^\infty \frac{1}{k^3+1}, \quad T_n = \sum_{k=n+1}^\infty \frac{1}{k^3} \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}.$$

Then $R_{100} \overset{(*)}{\leq} \int_{100}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(100)^2} = 0.00005.$]

Alternating Series and Absolute Convergence

An alternating series is a series whose terms are alternately positive and negative. The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test

- If $b_k > 0$, $b_k \geq b_{k+1}$ for all $k \geq 1$ and
- if $\lim_{k\to\infty} b_k = 0$,

then the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k-1}b_k = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad \text{is convergent.}$$

Furthermore,

- if $\sum_{k=1}^{\infty}(-1)^{k-1}b_k = s \in \mathbb{R}$, i.e. the alternating series converges to $s \in \mathbb{R}$, and
- if $R_n = s - s_n = \sum_{k=n+1}^{\infty}(-1)^{k-1}b_k$,

then for each $n = 1, 2, \ldots$ we have

$$\begin{aligned} |R_n| &= |s - s_n| \\ &= (b_{n+1} - b_{n+2}) + (b_{n+3} - b_{n+4}) + (b_{n+5} - b_{n+6}) + \cdots \\ &= b_{n+1} - (b_{n+2} - b_{n+3}) - (b_{n+4} - b_{n+5}) - \cdots \\ &\leq b_{n+1} \quad \text{since } b_{n+k} - b_{n+k+1} \geq 0 \text{ for all } k \geq 2. \end{aligned}$$

Examples

1. The alternating harmonic series $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$ is convergent by the Alternating Series Test.
2. The alternating harmonic series $\sum_{k=1}^{\infty}\frac{(-1)^{k}3k}{4k-1}$ is divergent since $\lim_{k\to\infty}\frac{3k}{4k-1} = \frac{3}{4} \neq 0$.
3. Determine the convergence of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k^2}{k^3+1}$.
4. Find the sum of the series $\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}$ correct to three decimal places.

   [Solution: First observe that the series $\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}$ is convergent by the Alternating Series Test. Since $b_7 = \frac{1}{7!} = \frac{1}{5040} < \frac{1}{5000} = 0.0002$ and $s_6 = \sum_{k=0}^{6}\frac{(-1)^{k}}{k!} \approx 0.368056$, so we have $s \approx 0.368$ correct to three decimal places.]

Definitions

- A series $\sum a_k$ is called absolutely convergent if the series of absolute values $\sum |a_k|$ is convergent.
- A series $\sum a_k$ is called conditionally convergent if it is convergent but not absolutely convergent; that is, if $\sum a_k$ converges but $\sum |a_k|$ diverges.

Theorem If a series $\sum a_k$ is absolutely convergent, then it is convergent.

Proof Since $\sum a_k$ is absolutely convergent and $|R_n| \leq \sum_{k=n+1}^{\infty} |a_k|$ for all n, we have $0 \leq \lim_{n\to\infty} |R_n| \leq \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |a_k| = 0 \implies \lim_{n\to\infty} |R_n| = 0$. Hence, the series $\sum a_k$ is convergent.

Examples

1. The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$ is absolutely convergent since $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k-1}}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2 > 1$).
2. The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ is conditionally convergent since $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ is convergent by the Alternating Series Test and $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k-1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent p-series.
3. Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

   [Solution: Since $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$ for all n and since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent by the Comparison Test, and hence it is convergent.]
4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$, (b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$, (c) $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2k+1}$

Definition By a rearrangement of an infinite series $\sum a_k$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\sum a_k$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{10} + a_6 + a_7 + a_{15} + \cdots$$

It turns out that if $\sum a_k$ is absolutely convergent series with sum s, then any rearrangement of $\sum a_k$ has the same sum s.

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$(*) \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = s = \ln 2,$$

where we assume that $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k}$ for $-1 < x \leq 1$. If we multiply by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}s = \frac{1}{2}\ln 2$$

Inserting zeros between the terms of this series, we have

$$(\dagger) \quad 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}s = \frac{1}{2}\ln 2$$

Now we add the series in $(*)$ and $(\dagger)$:

$$(**) \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{3}{2}s = \frac{3}{2}\ln 2$$

Notice that the series in $(**)$ contains the same terms as in $(*)$ but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

- if $\sum a_k$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_k$ that has a sum equal to r.

The Ratio and Root Tests

The Ratio Test Suppose that $a_k \neq 0$ for all $k = 1, 2, \ldots$.

(i) If $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L > 1$ or $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

(iii) If $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{k=1}^{\infty} a_k$.

Examples

1. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ for absolute convergence.

2. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n!}$.

3. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$.

4. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

The Root Test

(i) If $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = L < 1$, then the series $\displaystyle\sum_{k=1}^{\infty} a_k$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = L > 1$ or $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = \infty$, then the series $\displaystyle\sum_{k=1}^{\infty} a_k$ is divergent.

(iii) If $\displaystyle\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$, the Root Test is inconclusive.

Examples

1. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ converges or diverges.

Strategy for Testing Series

Examples In the following examples, don't work out all the details but simply indicate which tests should be used.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ [Solution: Use the Test for Divergence.]

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ [Solution: Use the Limit Comparison Test.]

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^4+1}$ [Solution: Use the Alternating Series Test. We can also observe that the series converges absolutely and hence converges.]

4. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$ [Solution: Use the Ratio Test.]

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ [Solution: Use the Comparison or the Limit Comparison Test.]

Power Series

Definition A power series in x is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k,$$

where x is a variable and the a_k's are constants called the coefficients of the series.

For each number that we substitute for x, the series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x.

The sum of the power series is a function

$$s(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$$

whose domain is the set of all x for which the series converges. Notice that $s(x)$ resembles a polynomial. The only difference is that $s(x)$ has infinitely many terms.

More generally, a series of the form

$$\sum_{k=0}^{\infty} a_k (x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x-a)^k$$

is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

Proposition Suppose that $a_k \neq 0$ for all $k = 1, 2, \ldots$, and for a fixed point $x \neq a$, suppose that

$$\lim_{k\to\infty} \sqrt[k]{|a_k(x-a)^k|} = \left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right) |x-a| = L < 1 \quad \text{or} \quad \left(\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}\right) |x-a| = L < 1.$$

Then $\sum_{k=1}^{\infty} a_k (y-a)^k$ is absolutely convergent for all $|y-a| \leq |x-a|$.

Theorem For a power series $\sum_{k=0}^{\infty} a_k (x-a)^k$, there are only three possibilities:

(1) The series converges only when $x = a$.

(2) The series converges for all x.

(3) There is a positive number R, called the radius of convergence of the power series, such that

- $\sum_{k=0}^{\infty} a_k (x-a)^k$ converges if $|x-a| < R$ and
- $\sum_{k=0}^{\infty} a_k (x-a)^k$ diverges if $|x-a| > R$.

By convention,

- the radius of convergence is $R = 0$ in case (1) and
- $R = \infty$ in case (2).

The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

- In case (1), the interval consists of just a single point a.
- In case (2), the interval is $(-\infty, \infty)$.
- In case (3), the interval is one of $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$ or $[a-R, a+R]$.

Proposition (radius of convergence) Suppose that $a_k \neq 0$ for all $k = 1, 2, \ldots$, and suppose that

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = L \quad \text{or} \quad \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L \quad \text{for some} \quad 0 \le L \le \infty.$$

Then the radius of convergence R of the power series $\sum_{k=1}^{\infty} a_k(x-a)^k$ is given by

(i) $R = 1/L$ if $0 < L < \infty$.
Proof Since

$$\lim_{k\to\infty} \sqrt[k]{|a_k(x-a)^k|} = \lim_{k\to\infty} \sqrt[k]{|a_k|}|x-a| \begin{cases} < L \cdot 1/L = 1 & \text{for each } |x-a| < R = 1/L, \\ > L \cdot 1/L = 1 & \text{for each } |x-a| > R = 1/L, \end{cases}$$

or

$$\lim_{k\to\infty} \frac{|a_{k+1}(x-a)^{k+1}|}{|a_k(x-a)^k|} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}|x-a| \begin{cases} < L \cdot 1/L = 1 & \text{for each } 0 < |x-a| < R = 1/L, \\ > L \cdot 1/L = 1 & \text{for each } |x-a| > R = 1/L, \end{cases}$$

so $\sum_{k=1}^{\infty} a_k(x-a)^k$

– converges for each $|x-a| < R = 1/L$,
– diverges for each $|x-a| > R = 1/L$,

and $R = 1/L$ is the radius of convergence of the power series $\sum_{k=1}^{\infty} a_k(x-a)^k$.

(ii) $R = \infty$ if $L = 0$.

(iii) $R = 0$ if $L = \infty$.

Examples

1. For what values of x is the series $\sum_{n=0}^{\infty} x^n$ convergent?

   [Solution: By the Ratio Test and the Divergence Test, the series converges (absolutely) only when $|x| < 1$.]

2. For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

   [Solution: By the Ratio Test, the Alternating Series Test and the p-series Test, the series converges only when $2 \le x < 4$.]

3. For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

   [Solution: By the Ratio Test, the series converges only when $x = 0$.]

4. For what values of x is the series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ convergent?

   [Solution: By the Ratio Test, the series converges (absolutely) when $x \in (-\infty, \infty)$.]

Representations of Functions as Power Series

Examples

1. Since the power series $\sum_{k=0}^{\infty} x^k$ converges absolutely for $|x| < 1$, and since

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1-x} \quad \text{for } |x| < 1,$$

we say that $\sum_{k=0}^{\infty} x^k$, is a power series representation of $\frac{1}{1-x}$ for $x \in (-1, 1)$.

2. Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

[Solution: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-x^2)^k$ converges for $|-x^2| < 1 \iff |x| < 1$.]

Differentiation and Integration of Power Series

Theorem If the power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{k=0}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

is differentiable (and therefore continuous) and

(i) $f'(x) = \frac{d}{dx} \sum_{k=0}^{\infty} a_k(x-a)^k = \sum_{k=0}^{\infty} \frac{d}{dx} \left[a_k(x-a)^k\right] = \sum_{k=1}^{\infty} k\, a_k(x-a)^{k-1}$

for each $x \in (a - R, a + R)$,

(ii) $\int f(x)\, dx = \int \sum_{k=0}^{\infty} a_k(x-a)^k\, dx = \sum_{k=0}^{\infty} \int a_k(x-a)^k\, dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}$

on the interval $(a - R, a + R)$.

(iii) the radii of convergence of $\sum_{k=1}^{\infty} k\, a_k(x-a)^{k-1}$ and $\sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}$ are both R,

(iv) f has derivatives of all order $n = 0, 1, 2\ldots$ on $(a-R, a+R)$ and for each $x \in (a-R, a+R)$,

$$f^{(n)}(x) = \frac{d^n}{dx^n} \sum_{k=0}^{\infty} a_k(x-a)^k = \sum_{k=0}^{\infty} \frac{d^n}{dx^n} \left[a_k(x-a)^k\right] = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)a_k(x-a)^{k-n}.$$

Examples

1. Express $\frac{1}{(1-x)^2}$ as a power series by differentiating the Equation $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. What is the radius of convergence?

[Solution: Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, and by differentiating both sides, we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n \quad \text{for } |x| < 1.$$

Since $\lim_{n\to\infty}(n+1)^{1/n} = 1$, the radius of convergence $R = 1$.]

2. Express $\tan^{-1} x$ as a power series by integrating the equation $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. What is the radius of convergence?

[Solution: For $|x| < 1$, since

$$\begin{aligned}\tan^{-1} x &= \tan^{-1} z|_0^x = \int_0^x \frac{1}{1+z^2}\,dz \\ &= \int_0^x \sum_{n=0}^{\infty}(-1)^n z^{2n}\,dz = \sum_{n=0}^{\infty}\int_0^x (-1)^n z^{2n}\,dz = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\end{aligned}$$

and since $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$ converges when $x = \pm 1$, we have

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \quad \text{for all } |x| \le 1.]$$

Examples (from Section 17.4)

1. Use power series $y = \sum_{n=0}^{\infty} c_n x^n$ to solve the equation $y' = ry$, where r is a constant.

$$\begin{aligned} 0 &= y' - ry = \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} rc_n x^n = \sum_{n=0}^{\infty}[(n+1)c_{n+1} - rc_n]x^n \\ \Longrightarrow \quad & c_{n+1} = \frac{rc_n}{n+1} \quad \text{for } n = 0, 1, 2, \ldots \text{ (called a recursive relation)} \\ \Longrightarrow \quad & c_{n+1} = \frac{rc_n}{n+1} = \frac{r^2 c_{n-1}}{(n+1)n} = \cdots = \frac{r^{n+1}c_0}{(n+1)!} \\ \Longrightarrow \quad & y = c_0 \sum_{n=0}^{\infty}\frac{r^n}{n!}x^n = c_0 e^{rx}\end{aligned}$$

2. Use power series $y = \sum_{n=0}^{\infty} c_n x^n$ to solve the equation $y'' + ry = 0$, where $r > 0$ is a constant.

$$\begin{aligned} 0 &= y'' + ry = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} rc_n x^n = \sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + rc_n]x^n \\ \Longrightarrow \quad & c_{n+2} = \frac{-rc_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, \ldots \text{ (called a recursive relation)}\end{aligned}$$

$$\Longrightarrow \quad c_{2n} = \frac{(-r)c_{2n-2}}{(2n)(2n-1)} = \frac{(-r)^2 c_{2n-4}}{(2n)(2n-1)(2n-2)(2n-3)} = \cdots = \frac{(-r)^n c_0}{(2n)!}, \quad n = 0, 1, 2, \ldots$$

$$c_{2n+1} = \frac{(-r)c_{2n-1}}{(2n+1)(2n)} = \frac{(-r)^2 c_{2n-3}}{(2n+1)(2n)(2n-1)(2n-2)} = \cdots = \frac{(-r)^n c_1}{(2n+1)!}, \quad n = 0, 1, 2, \ldots$$

$$\Longrightarrow \quad y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n r^n}{(2n)!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n r^n}{(2n+1)!} x^{2n+1} = c_0 \cos \sqrt{r}x + c_1 \sin \sqrt{r}x$$

3. Show that $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, the Bessel function of order 0, is a solution of the Bessel equation $x^2y'' + xy' + x^2y = 0$.

If $y = \sum_{n=0}^{\infty} c_n x^n$, then $x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2}$, and

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$\Longrightarrow \quad xy' = \sum_{n=1}^{\infty} nc_n x^n, \quad x^2y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^n$$

$$\Longrightarrow \quad xy' = c_1 x + \sum_{n=2}^{\infty} nc_n x^n, \quad x^2y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^n$$

$$\overset{\text{set } k=n-2}{\underset{n=k+2}{\Longrightarrow}} \quad xy' = c_1 x + \sum_{k=0}^{\infty} (k+2)c_{k+2} x^{k+2}, \quad x^2y'' = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^{k+2}$$

$$\overset{\text{set } k=n}{\Longrightarrow} \quad xy' = c_1 x + \sum_{n=0}^{\infty} (n+2)c_{n+2} x^{n+2}, \quad x^2y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^{n+2}$$

$$\Longrightarrow \quad 0 = x^2y'' + xy' + x^2y = c_1 x + \sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+2)c_{n+2} + c_n] x^{n+2}$$

$$\Longrightarrow \quad 0 = c_1 x + \sum_{n=0}^{\infty} [(n+2)^2 c_{n+2} + c_n] x^{n+2}$$

$$\Longrightarrow \quad c_1 = 0, \quad c_{n+2} = \frac{(-1)c_n}{(n+2)^2} \quad \text{for } n = 0, 1, 2, \ldots \text{ (called a recursive relation)}$$

$$\Longrightarrow \quad c_{2n+1} = 0, \; c_{2n} = \frac{(-1)c_{2n-2}}{(2n)^2} = \frac{(-1)^2 c_{2n-4}}{[2^2n^2][2^2(n-1)^2]} = \cdots = \frac{(-1)^n c_0}{2^{2n}(n!)^2} \quad \text{for } n = 0, 1, 2, \ldots$$

$$\Longrightarrow \quad y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \overset{\text{set } c_0=1}{=} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = J_0(x).$$

(a) Find the domain of $J_0(x)$. [Solution: By the Ratio Test, the series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty, \infty)$.]

(b) Find the derivative of $J_0(x)$. [Solution: $J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2}$.]

Taylor and Maclaurin Series

Taylor's Theorem If $f(x)$ has derivatives of all orders in an open interval $I=(a-R,a+R)$ containing a, then for each positive integer n and for each $x\in I$,

$$f(x)=P_n(x)+R_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{1}{n!}\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt,$$

where $f^{(k)}(a)=\dfrac{d^kf}{dx^k}(a)$ is the k^{th} derivative of f at a for $k\ge 1$, $f^{(0)}(a)=f(a)$, and

- $P_n(x)=\displaystyle\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{1}{n!}\frac{d^nf}{dx^n}(a)(x-a)^n$ is called the n^{th}-degree Taylor polynomial of f at a
- $R_n(x)=f(x)-P_n(x)=\dfrac{1}{n!}\displaystyle\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt$ is called the remainder of order n for the approximation of $f(x)$ by $P_n(x)$ over I.

Proof Using integration by parts formula $\displaystyle\int_a^x u\,dv=uv|_a^x-\int_a^x v\,du$ repeatedly, we get

$$\begin{aligned}
f(x)-f(a) &= \int_a^x f'(t)\,dt=-\int_a^x f'(t)\,d(x-t), \quad u=f'(t),\ dv=-d(x-t)\\
&= -f'(t)(x-t)|_a^x+\int_a^x f''(t)\,(x-t)\,dt\\
&= f'(a)(x-a)-\frac{1}{2!}\int_a^x f''(t)\,d(x-t)^2, \quad u=f''(t),\ dv=-\frac{d(x-t)^2}{2!}\\
&= f'(a)(x-a)-\frac{1}{2!}f''(t)(x-t)^2|_a^x+\frac{1}{2!}\int_a^x f'''(t)(x-t)^2\,dt\\
&= f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2-\frac{1}{3!}\int_a^x f'''(t)\,d(x-t)^3\\
&\cdots\quad\cdots\quad\cdots\\
&\overset{(*)}{=} f'(a)(x-a)+\cdots+\frac{1}{n!}f^{(n)}(a)(x-a)^n+\frac{1}{n!}\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt
\end{aligned}$$

The Remainder Estimation Theorem If there exists a positive constant such that $|f^{(n+1)}(x)|\le M$ for all $|x-a|\le R$, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}\quad\text{for } |x-a|\le R.$$

Proof Taylor's Theorem implies that

$$|R_n(x)|\overset{(*)}{=}|f(x)-P_n(x)|\le\frac{1}{n!}\left|\int_a^x |f^{(n+1)}(t)|\,(x-t)^n\,dt\right|\le\frac{M}{(n+1)!}\left|(x-t)^{n+1}\,|_a^x\right|=\frac{M}{(n+1)!}|x-a|^{n+1}.$$

Lagrange Remainder In fact, since $f^{(n+1)}(x)$ is continuous for each $|x-a|<R$ and by the Mean Value Theorem for an integral, there exists a c between a and x such that

$$\begin{aligned}
R_n(x) &\overset{(*)}{=} f(x)-P_n(x)=\frac{1}{n!}\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt=\frac{-1}{(n+1)!}\int_a^x f^{(n+1)}(t)\,d(x-t)^{n+1}\\
&= \frac{-f^{(n+1)}(c)}{(n+1)!}\int_a^x d(x-t)^{n+1}\,dt=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.
\end{aligned}$$

Theorem If $\lim_{n\to\infty} R_n(x) = 0$ for each $|x-a| < R$, then f has a power series expansion at a, that is

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad \text{for each } |x-a| < R,$$

where the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the Taylor series of the function f at a (or about a or centered at a).

In case $a = 0$, the Taylor series becomes

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \quad \text{for } |x| < R$$

and it is called the Maclaurin series of f.

Examples

1. Find the Maclaurin series of the function $f(x) = \frac{1}{1 \mp x}$ and its radius of convergence.
   [Solution: $\frac{1}{1 \mp x} = \sum_{n=0}^{\infty}(\pm x)^n$ and $R = 1$ by the Ratio Test.]
2. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.
   [Solution: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ and $R = \infty$ by the Ratio Test.]
3. Find the Maclaurin series of the function $f(x) = \sin x$ and its radius of convergence.
   [Solution: $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $R = \infty$ by the Ratio Test.]
4. Find the Maclaurin series of the function $f(x) = \cos x$ and its radius of convergence.
   [Solution: $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ and $R = \infty$ by the Ratio Test.]
5. Find the Maclaurin series of the function $f(x) = \tan^{-1} x$ and its radius of convergence.
   [Solution: $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ and $R = 1$ by the Ratio Test.]
6. Find the Maclaurin series of the function $f(x) = \ln(1+x)$ and its radius of convergence.
   [Solution: $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}$ and $R = 1$ by the Ratio Test.]
7. Find the Maclaurin series of the function $f(x) = (1+x)^k$ and its radius of convergence.
   [Solution: $(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$ and $R = 1$ by the Ratio Test.]
8. Find the Maclaurin series for (a) $f(x) = x\cos x$ and (b) $f(x) = \ln(1+3x^2)$.
   [Solution: $x\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$ and $\ln(1+3x^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n x^{2n}}{n} = \sum_{k=0}^{\infty}(-1)^k \frac{3^{k+1}x^{2k+2}}{k+1}$]

9. Find the first three nonzero terms in the Maclaurin series for $(a)\ e^x \sin x$ and $(b)\ \tan x$.
[Solution: $(a)\ x + x^2 + \frac{1}{3}x^3 + \cdots$; $(b)\ x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$]

Examples

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
[Solution: $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.]

(b) How accurate is this approximation when $7 \le x \le 9$?
[Solution: Because $x \ge 7$, we have $x^{8/3} \ge 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \le \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021 \implies |R_2(x)| \le \frac{0.0021}{3!}|x-8|^3 < 0.0004.\ ]$$

Some Proofs

(a) **(The Direct Comparison Test)** Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series with positive terms. If $\sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} b_k$ is convergent and $a_k \le b_k$ for all k, then $\sum_{k=1}^{\infty} a_k$ is also convergent.

Proof Since $\sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} b_k$ is convergent,

$$0 = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} b_k \ge \lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k \ge 0 \implies \lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k = 0 \quad \text{by the squeeze theorem,}$$

and that $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ is convergent.

(b) **(The Limit Comparison Test)** Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series with positive terms.

- If $\lim_{k\to\infty} \frac{a_k}{b_k} = r \in (0,\infty)$, then either both series converge or both diverge.
  Proof Let $\varepsilon = \frac{r}{2} > 0$. Since $\lim_{k\to\infty} \frac{a_k}{b_k} = r \in (0,\infty)$, there is an $M \in \mathbb{N}$ such that

$$\begin{aligned} & \left|\frac{a_k}{b_k} - r\right| < \varepsilon = \frac{r}{2} \quad \text{for all } k \ge M \\ \iff & -\frac{r}{2} < \frac{a_k}{b_k} - r < \frac{r}{2} \iff \frac{r}{2} < \frac{a_k}{b_k} < \frac{3r}{2} \quad \text{for all } k \ge M \\ \iff & \frac{r}{2} b_k < a_k < \frac{3r}{2} b_k \quad \text{for all } k \ge M. \\ \implies & 0 < \frac{r}{2}\sum_{k=n}^{\infty} b_k < \sum_{k=n}^{\infty} a_k < \frac{3r}{2}\sum_{k=n}^{\infty} b_k \quad \text{for all } n \ge M. \end{aligned}$$

- If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and if $\sum_{k=1}^{\infty} b_k$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is convergent.

Proof Since $\lim_{k\to\infty}\frac{a_k}{b_k} = 0$, there is an $M \in \mathbb{N}$ such that

$$0 < \frac{a_k}{b_k} = \left|\frac{a_k}{b_k} - 0\right| < 1 \quad \text{for all } k \geq M \iff 0 < \frac{a_k}{b_k} < 1 \quad \text{for all } k \geq M$$

$$\iff 0 < a_k < b_k \quad \text{for all } k \geq M. \implies 0 < \sum_{k=n}^{\infty} a_k < \sum_{k=n}^{\infty} b_k \quad \text{for all } n \geq M.$$

- If $\lim_{k\to\infty}\frac{a_k}{b_k} = \infty$ and if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} b_k$ is convergent.

  Proof Since $\lim_{k\to\infty}\frac{a_k}{b_k} = \infty$, there is an $M \in \mathbb{N}$ such that $\frac{a_k}{b_k} > 1$ for all $k \geq M$. Thus we have

$$a_k > b_k \quad \text{for all } k \geq M \implies \sum_{k=n}^{\infty} a_k > \sum_{k=n}^{\infty} b_k > 0 \quad \text{for all } n \geq M.$$

(c) **(The Ratio Test)** Suppose that $a_k \neq 0$ for all $k = 1, 2, \ldots$, and suppose that $\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (and therefore convergent).

Proof Given $\frac{1-L}{2} > \varepsilon > 0$, since $\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = L < 1$, there is an $M \in \mathbb{N}$ such that

$$\begin{aligned}
&\left|\frac{|a_{k+1}|}{|a_k|} - L\right| < \varepsilon < \frac{1-L}{2} \quad \text{for all } k \geq M \\
\implies &\frac{|a_{k+1}|}{|a_k|} < L + \frac{1-L}{2} = \frac{1+L}{2} < 1 \quad \text{for all } k \geq M \\
\implies &|a_{k+1}| < \left(\frac{1+L}{2}\right)|a_k| \quad \text{for all } k \geq M \\
\implies &|a_k| < \left(\frac{1+L}{2}\right)|a_{k-1}| < \left(\frac{1+L}{2}\right)^2|a_{k-2}| < \cdots < \left(\frac{1+L}{2}\right)^{k-M}|a_M| \quad \text{for all } k \geq M \\
\implies &\sum_{k=n}^{\infty}|a_k| \leq \sum_{k=n}^{\infty}\left(\frac{1+L}{2}\right)^{k-M}|a_M| = \left(\frac{1+L}{2}\right)^{n-M}\cdot\frac{|a_M|}{1-(1+L)/2} \quad \text{for all } n \geq M
\end{aligned}$$

(d) **(The Root Test)** If $\lim_{k\to\infty}\sqrt[k]{|a_k|} = L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (and therefore convergent).

Proof Given $\frac{1-L}{2} > \varepsilon > 0$, since $\lim_{k\to\infty}\sqrt[k]{|a_k|} = L < 1$, there is an $M \in \mathbb{N}$ such that

$$\begin{aligned}
&\left|\sqrt[k]{|a_k|} - L\right| < \varepsilon \quad \text{for all } k \geq M \\
\implies &\sqrt[k]{|a_k|} - L < \varepsilon < \frac{1-L}{2} \quad \text{for all } k \geq M \\
\implies &0 \leq \sqrt[k]{|a_k|} \leq \left(\frac{1+L}{2}\right) < 1 \quad \text{for all } k \geq M \\
\implies &|a_k| \leq \left(\frac{1+L}{2}\right)^k \quad \text{for all } k \geq M
\end{aligned}$$

$$\Longrightarrow \quad \sum_{k=n}^{\infty} |a_k| \le \sum_{k=n}^{\infty} \left(\frac{1+L}{2}\right)^k = \left(\frac{1+L}{2}\right)^n \cdot \frac{1}{1-(1+L)/2} \quad \text{for all } n \ge M$$

(e) **(Lagrange Remainder)** If $f^{(n+1)}(t)$ is continuous on $[a,x]$, then there exists a $c \in [a,x]$ such that

$$\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt = f^{(n+1)}(c) \int_a^x (x-t)^n\,dt = \frac{f^{(n+1)}(c)}{n+1}(x-a)^{n+1}.$$

Proof Since $f^{(n+1)}(t)$ is continuous and $(x-t)^n \ge 0$ on $[a,x]$, there exist m and M such that $m \le f^{(n+1)}(t) \le M$ for each $t \in [a,x]$ and

$$\begin{aligned} & m \int_a^x (x-t)^n\,dt \le \int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt \le M \int_a^x (x-t)^n\,dt \\ \Longrightarrow \quad & m \le \frac{\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt}{\int_a^x (x-t)^n\,dt} \le M. \end{aligned}$$

By the Intermediate Value Theorem, there is a point $c \in [a,x]$ such that

$$f^{(n+1)}(c) = \frac{\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt}{\int_a^x (x-t)^n\,dt} \Longrightarrow \int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt = \frac{f^{(n+1)}(c)}{n+1}(x-a)^{n+1}.$$